CHARGES AND FIELDS IN A CURRENT-CARRYING WIRE

Dragan V Redžić
Faculty of Physics, University of Belgrade, PO Box 368, 11001 Beograd, Serbia, Yugoslavia
E-mail address: redzic@ff.bg.ac.rs

Abstract
Charges and fields in a straight, infinite, cylindrical wire carrying a steady current are determined in the rest frames of ions and electrons, starting from the standard assumption that the net charge per unit length is zero in the lattice frame and taking into account a self-induced pinch effect. The analysis presented illustrates the mutual consistency of classical electromagnetism and Special Relativity. Some consequences of the assumption that the net charge per unit length is zero in the electrons frame are also briefly discussed.

1. Introduction

As is well known, combining Coulomb's law, charge invariance and the transformation law of a *pure* relativistic three-force [1,2], one can derive the correct equation for the force with which a point charge in uniform motion acts on any other point charge in *arbitrary* motion, and thus recognize both the **E** and **B** fields of a uniformly moving point charge and the corresponding Lorentz force expression [3-5]. Thus one can prove *indirectly*, without introducing general transformations for **E** and **B**, that the Lorentz force expression, $\mathbf{f}_L \equiv q\mathbf{E} + q\mathbf{u}\times\mathbf{B}$, transforms in the same way as the time derivative of the relativistic momentum of a particle with *time independent* mass, in the special case of **E** and **B** due to a uniformly moving point charge. Following the same line of reasoning, the so-called relativistic nature of the magnetic field is often illustrated by discussing the force on a charged particle outside a current-carrying wire, or the force between two parallel current-carrying wires [5-8]. (Note that in the latter case, contrary to the widespread opinion,

both electrostatics and magnetostatics need to be retained in the rest frame of electrons, as van Kampen [9] pointed out, though not in an irreproachable way [10,11].)

The above-mentioned arguments are intended to provide a short cut towards classical electromagnetism via relativity. However, those pedagogical vehicles, while ingenious, can be somewhat opaque to the student. As can be seen, they are based on two mighty tacit assumptions, namely, that Maxwell's equations are Lorentz covariant and also that electromagnetic force on a point charge (which is the Coulomb force in the right cases) transforms in the same way as $d(m\mathbf{u}/\sqrt{1-u^2/c^2})/dt$ with $m = \text{const}$; unfortunately, to prove these assumptions without using four-vectors is a real *tour de force* ([12-14], cf also [2]). Thus, the familiar derivations that are aimed at reaching electromagnetism from Coulomb's law (electrostatics) and relativity seem to be little more than simple illustrations of the mutual consistency of classical electromagnetism and Special Relativity (cf [15]).

An essential part in those pedagogical discussions, as well as in related discussions of the transformation laws for $\mathbf{E}$ and $\mathbf{B}$ (cf, e.g., [6]), is played by a long conducting wire carrying a steady current. The wire is usually modeled as consisting of two superposed lines of charge, one moving and an oppositely charged one at rest, extending along the z axis. For the convenience of the reader, we discuss briefly a typical example that illustrates simply the consistency of electromagnetism and relativity.

In the laboratory frame S, suppose there is a line of positive charge at rest with constant linear charge density λ_+, and a line of negative charge with density $\lambda_- = -\lambda_+$ moving to the right with a constant velocity $\mathbf{v} = v\mathbf{u}_z$. Since the net charge density is always zero, the electric field vanishes in the S frame. However, due to a current $I = |\lambda_-|v = \lambda_+ v$ to the left, there is the magnetic field whose direction is azimuthal around the line charges with sense given by the right-hand rule; from Ampère's law, $B = \mu_0 I/2\pi r$, where r is the distance from the line charges. A test charge q placed at rest in S remains at rest since there is no electric force on it (the electric field is zero) and no magnetic force (the test charge is at rest).

Consider now the same situation in the rest frame of the negative charges, S'. In that frame, the positive charges move to the left with velocity $-\mathbf{v}$. The corresponding line charge densities are $\lambda'_+ = \gamma\lambda_+$ and $\lambda'_- = -\lambda_+/\gamma$, where $\gamma \equiv (1 - v^2/c^2)^{-1/2}$, due to the Lorentz contraction and charge invariance. Then $\lambda' = \lambda'_+ + \lambda'_- = \lambda_+\gamma v^2/c^2$ and using Gauss' law or otherwise ([5], [16,17], cf also [18]), we find that there is a radial electric field in S', pointing away from the line charges, $E'_r = (1/2\pi\varepsilon_0 r)\lambda_+\gamma v^2/c^2$. Also, due to the motion of the positive charges, there is an azimuthal magnetic field in S'; using Ampère's law we find that $B' = \mu_0 I'/2\pi r$, where $I' = \gamma\lambda_+ v = \gamma I$ is the current in S'. The $\mathbf{E}'$ and $\mathbf{B}'$ fields can also be obtained directly, by Lorentz-transforming the corresponding $\mathbf{E}$ and $\mathbf{B}$ fields in S. The test charge q, at rest in S, moves uniformly with velocity $-\mathbf{v}$ relative to S', consistent with the fact that the corresponding Lorentz force, $q\mathbf{E}' + q(-\mathbf{v}) \times \mathbf{B}'$, vanishes (and also consistent with the transformation law of a pure relativistic three-force).

Some time ago, Peters [19] pointed out that the importance of Special Relativity would be made more evident if in the above example one used a more realistic model of a conducting wire. The author considered an infinite cylindrical conductor of circular cross section, whose material was idealized as consisting of the lattice of immovable positive ions at rest in the lab frame S, and an equal number of electrons which are free to move through the lattice. The permittivity and permeability of such a material are taken to be equal to those of free space, ε_0 and μ_0. When there is a steady current in the conductor, the free-electrons have a nonzero average velocity $\mathbf{v} = v\mathbf{u}_z$ in the rest frame of the lattice S ("the drift velocity"). In the S' frame, the free-electrons are, on average, at rest, and the lattice moves to the left with velocity $-\mathbf{v}$.

Now some simple questions arise. If one assumes that the infinite wire carrying a steady current is electrically neutral in the lattice frame S, is it possible to give an analysis of fields and forces in the S and S' frames consistent with classical electromagnetism and Special Relativity? Are the results obtained for the more realistic model of the infinite conducting wire analogous to those derived for the system consisting of superposed lines of

charge?

Peters came to a somewhat surprising conclusion that the bulk of the conductor was neutral with respect to the rest frame of the free-electrons S' and *not* with respect to the lattice frame S. Moreover, the author argued that the assumption of *overall* neutrality of the wire in the S frame led to a contradiction. Namely, according to Peters, if the wire is neutral in S then there should exist a positive surface charge density on the wire, generated by a self-induced pinch effect of the free-electrons. A surface charge in S implies the corresponding surface charge in S', due to charge invariance. However, Peters claimed in [19] that no mechanism for generating a surface charge existed in S', and thus the assumption of overall neutrality in S was questionable. His argument was criticized by Hernández *et al* [20] who pointed out that there is a mechanism for generating surface charge in S' too. Gabuzda [21] proposed such a mechanism and calculated the volume, surface and linear charge densities in the S and S' frames.

It seems, however, that Peters' analysis of the problem contains a difficulty with the concept of surface charge that went unnoticed by the authors of References [20,21]. Also, Gabuzda's discussion [21] seems to be based on a problematic starting assumption, which appears occasionally in the literature. In this paper, we attempt to give an analysis free from contradictions of charges and fields of an infinite current-carrying wire modeled as in Reference [19], both in the ions and electrons rest frames. The analysis leads to some interesting insights and, hopefully, could be an intriguing reading for the student of relativistic electrodynamics, together with our recent contributions to the subject [2,22].

2. Charges and fields in the lattice frame

Consider a straight, infinite, cylindrical conductor having a circular cross section of radius a, the axis of the conductor coinciding with the z axis. The lattice consists of uniformly distributed positive ions which are immovable; thus the volume charge density of the lattice ions, ρ_+, is spatially and temporally constant. When there is no current in the wire, the drift velocity of the free-electrons, the electric field inside the conductor, and the magnetic field

everywhere are all zero. From Gauss' law it follows that the volume charge density of the free-electrons is $-\rho_+$. If the conductor is overall neutral, then there is no surface charge over its surface, if it is sufficiently far from other bodies.

Consider now the case when there is a steady current in the wire to the left and assume, as is usually done, that all the free-electrons have equal axial drift velocity $\mathbf{v} = v\mathbf{u}_z$ to the right. The corresponding current density $\mathbf{J}$ is purely axial and, by symmetry, it can be expressed as a function of a single variable r, denoting distance from the axis of the conductor. Then, according to Ohm's law, there exists inside the conductor an axial electrostatic field $\mathbf{E}_\parallel = E_\parallel(-\mathbf{u}_z)$ which, by symmetry, depends only on r as well. The current produces a magnetic field whose lines are circles around the conductor's axis, with sense given by the right-hand rule. Consequently, there is a magnetic force on the free-electrons directed inward. Thus, in a steady configuration, there must exist also a transverse electric field $\mathbf{E}_\perp$ inside the wire, directed inward, satisfying relation

$$\mathbf{E}_\perp = -\mathbf{v} \times \mathbf{B}\,, \tag{1}$$

where $\mathbf{B}$ is the magnetic flux density inside the conductor due to the current in the conductor. Stress that Equation (1) applies at the points inside the conductor where $\mathbf{J} \neq 0$, i. e. where the charge density of the free-electrons is not zero. (Recall that an analogous situation arises in the Hall effect, only in that case $\mathbf{B}$ is not due to the current itself but represents an externally applied magnetic field.)

Now we shall determine the steady-state distribution of the free-electrons inside the current-carrying wire. Using Gauss' law, $\rho = \varepsilon_0 \nabla \cdot \mathbf{E}$, and Equation (1), we find that the net charge density inside the conductor, at the points where Equation (1) applies, satisfies equation

$$\rho = \varepsilon_0 \nabla \cdot \mathbf{E}_\perp = \varepsilon_0 \mathbf{v} \cdot (\nabla \times \mathbf{B})\,, \tag{2}$$

since $\nabla \cdot \mathbf{E}_\parallel = 0$ and $\mathbf{v}$ is constant. As ρ and $\mathbf{J}$ are stationary, Ampère's law applies

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} = \mu_0 \rho_- \mathbf{v}\,, \tag{3}$$

where ρ_- is the charge density of the free-electrons in the steady state. Thus

$$\rho = \varepsilon_0 \mu_0 \rho_- v^2 \equiv \rho_- v^2/c^2\,. \tag{4}$$

Eventually, since $\rho = \rho_+ + \rho_-$, from Equation (4) we get

$$\rho_- = -\rho_+ \gamma^2\,. \tag{5}$$

Equation (5) implies that ρ_- is spatially and temporally constant, because ρ_+ is constant, and also that $|\rho_-| > \rho_+$; thus, $\mathbf{J}$ is constant too. A simple calculus, starting from $\nabla \times \mathbf{E} = 0$ and Equation (1), reveals that $\mathbf{E}_\parallel$ is constant as well, which, in combination with previous results and Ohm's law, implies that all the free-electrons have equal mobilities. (Note that in the problem considered, the usual statement of Ohm's law, $\mathbf{J} = \sigma\mathbf{E}$, where σ is the conductivity of the material, need to be replaced by a more general form, $\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, which reduces to $\mathbf{J} = \sigma\mathbf{E}_\parallel$.) Equations (4) and (5) were obtained using Gauss' law and Ampère's law in their integral forms by Matzek and Russell [23]; the differential approach used above is adapted from Rosser [24].

Greater density of the free-electrons when they are in motion was explained by previous authors [23,19,21] by a self-induced pinch effect of the electrons. Namely, during the transient build-up of the current in the wire, a magnetic pinch will develop as the electrons in the wire are macroscopically accelerated (i. e. as their mean velocity varies from zero up to the steady drift velocity $\mathbf{v}$). Consequently, when equilibrium is reached, the free-electrons are concentrated towards the axis of the wire, leaving a region at the conductor's surface which is swept clear of the electrons by this pinch effect, as was pointed out in [23]. Thus, there remains a layer of positive ions at the surface, whose thickness δ can be determined from the assumption that the wire is neutral also when it carries a steady current. Is that assumption plausible?

We are faced here with a tricky problem of how is a steady current established in the conducting wire. It seems that there are two solutions to the problem, depending on initial conditions; this will be discussed in some detail in the last section. Now, as is usually done for teaching purposes, we shall find the thickness of the "naked" layer of positive ions starting from the standard assumption that the current in the wire is established in such a way that the net charge per unit length of the wire is zero in the lattice frame. (Recall that the standard argument in "current without pinch" discussions claims that "the only difference between a wire carrying a current and a wire not carrying a current is the existence of a drift velocity for the electrons. The mean distance between the electrons remains unaffected as measured in the laboratory frame", [5, p 259], cf also [18], and the last section.)

That assumption obviously implies

$$\rho_+ a^2 \pi l + \rho_- (a-\delta)^2 \pi l = 0\,, \tag{6}$$

where l is the length of a segment of the conductor between two parallel planes which are perpendicular to its axis. From Equations (5) and (6) one has

$$a - \delta = a\sqrt{1 - v^2/c^2} \equiv a/\gamma\,. \tag{7}$$

The electron drift velocities are typically fractions of millimeter per second so that the ratio v/c is about 10^{-12}, and the thickness of the naked layer δ is "almost inconceivably small". It is therefore tempting to assume that the thickness δ may be neglected in calculations for all reasonable values of a and introduce the corresponding surface charge density. So some authors [19,21] take that, in a steady configuration, Equations (4) and (5) are valid for all $r < a$ and calculate the corresponding surface charge density ς from the condition

$$(\rho_+ + \rho_-) a^2 \pi l + \varsigma 2\pi a l = 0\,,$$

which gives $\varsigma = |\rho| a/2$. However, in this way extra free-electrons of charge

$\rho_-[a^2-(a-\delta)^2]\pi l$ and an opposite extra positive charge $\varsigma 2\pi al$, have been created *ex nihilo* and added to the existing charge distribution in the wire segment, described by Equations (4)-(7). As can be seen, the amounts of these fictitious extra charges are comparable to those of the existing charges in the wire segment. Consequently, it appears that the thickness of the naked layer δ must not be neglected and thus the concept of surface charge should be discarded in this case. It can be methodologically dangerous to neglect v^2/c^2 at one point of analysis, and to retain the same quantity elsewhere. (True, there is a way to save the surface charge concept, altering *distances* but not *charges*. Thus, one could argue that the correct surface charge density, ς^*, is found from the condition $\rho_+[a^2-(a-\delta)^2]\pi l = \varsigma^* 2\pi al$, and the volume charge densities ρ_+ and ρ_- should be modified accordingly. For example, the corresponding positive volume charge density would be given by $\rho_+^*\pi a^2 l = \rho_+(a-\delta)^2\pi l$. However, the above procedure gives an approximate solution to the problem which is, to say the least, more complicated than the correct one.)

From the preceding considerations it follows that in the present analysis we should retain δ everywhere and thus, somewhat surprisingly, only volume and linear densities of charge should be used.

It is now simple to determine the electric and magnetic fields in the S frame using $\nabla\times\mathbf{E}=0$, Gauss' law, Ampère's law, and Ohm's law, and taking into account that the current I in the conductor is uniformly distributed inside the circle of radius $r=a-\delta$. For the sake of completeness, we present the final results for $\mathbf{E}$ and $\mathbf{B}$.

The magnitude of the magnetic flux density is given by

$$B = \begin{cases} \mu_0 Ir/2\pi(a-\delta)^2\,, & r \le a-\delta\,, \\ \mu_0 I/2\pi r\,, & r > a-\delta\,. \end{cases} \tag{8}$$

The axial electric field is constant *everywhere*

$$\mathbf{E}_{\|} = -I\mathbf{u}_z/\pi(a-\delta)^2\sigma\,, \tag{9}$$

and the transverse (radial) electric field can be expressed as

$$\mathbf{E}_\perp = \begin{cases} -\mu_0 I v r \mathbf{u}_r / 2\pi (a-\delta)^2\,, & r \leq a-\delta\,, \\ -\mu_0 I v (a^2 - r^2)\mathbf{u}_r / 2\pi r a^2 (v^2/c^2)\,, & a-\delta \leq r \leq a\,, \\ 0\,, & r > a\,, \end{cases} \tag{10}$$

using relation

$$I = \rho_+ \pi a^2 v\,. \tag{11}$$

If there is a test charge q near the wire which is instantaneously at rest relative to the S frame, then the force $\mathbf{F} = q\mathbf{E}_\parallel$ acts on the charge at that moment. Next, if $q > 0$, the charge would deflect towards the wire due to the combined action of $\mathbf{E}_\parallel$ and $\mathbf{B}$.

3. Charges and fields in the free-electrons frame

Consider now the same current-carrying wire in the steady-state configuration as observed in the S' frame, which moves with velocity $\mathbf{v} = v\mathbf{u}_z$ relative to the S frame. In S', the free-electrons are, on average, at rest, and the positive lattice ions move to the left with velocity $-\mathbf{v}$. One easily finds that $\rho'_+ = \gamma\rho_+$ and $\rho'_- = \rho_-/\gamma = -\gamma\rho_+$, due to the Lorentz contraction, charge invariance and Equation (5). Since transverse lengths are Lorentz invariant, the radius of the wire is still a, and the free-electrons are still inside the cylinder of radius $a-\delta$. Thus, the bulk of the conductor ($r < a - \delta$) is neutral in the S' frame! However, the net charge per unit length of the conductor, λ', is not zero, due to the naked layer of moving positive ions ($a - \delta < r < a$). Obviously,

$$\lambda' = \rho'_+ \pi [a^2 - (a-\delta)^2] = \gamma \rho_+ \pi a^2 v^2/c^2\,, \tag{12}$$

using relation $\rho'_+ = \gamma\rho_+$, Equation (7) and identity $\gamma^2 - 1 \equiv \gamma^2 v^2/c^2$. Equation (12) is consistent with the corresponding equation for the simple system of two superposed lines of charge, discussed in the Introduction, since $\rho_+ \pi a^2 \equiv \lambda_+$. (Note that the result for λ' derived in Reference [21], Equation (5) in [21], which in our notation reads $\lambda' = \gamma^3 \rho_+ \pi a^2 v^2/c^2$, is different from our result (12) due to the contribution of the fictitious surface charges introduced in [21].)

The thickness δ of the naked positive layer in S' is deduced above from Lorentz invariance of transverse lengths. The same result is obtained by applying charge conservation in S'. Namely, in the "initial", "current without pinch", situation in S', the lattice shrinks along the direction of motion whereas the axial expansion of the free-electrons occurs, due to the Lorentz contraction, as compared with the corresponding "current without pinch" situation as observed in the S frame. Since the ions and electrons charge densities in S are then ρ_+ and $-\rho_+$ *everywhere*, the corresponding S' densities are $\rho_+\gamma$ and $-\rho_+/\gamma$ throughout the wire. Consequently, there is a radial electric field directed outward in the wire in S' due to an excess of positive charge, and the free-electrons move inward until the net charge density is zero in the coaxial cylindrical region enriched by the electrons and the radial electric field vanishes, as was suggested by Matzek and Russell [23]. Thus, charge conservation applied to a wire segment of length l in S' gives

$$[\rho_+\gamma + (-\rho_+/\gamma)]a^2\pi l = \rho_+\gamma a^2\pi l + (-\rho_+\gamma)(a-\delta)^2\pi l\,, \tag{13}$$

wherefrom we find that δ satisfies Equation (7), as it should.

The fact that the net linear charge density is zero in S and nonzero in S' might appear at first sight contradictory to charge invariance; but the paradox is only apparent. Namely, as can be seen, for a well-defined wire segment in the lattice frame S (the positive ions at rest and an identical number of moving free-electrons taking up the same length d in S at one instant of the S time), there is no corresponding *wire segment* in S' because these ions and electrons take up different lengths in S' at a certain instant of the S' time, d/γ and $d\gamma$, respectively. In other words, wire segments cannot be reified, the ions and the corresponding electrons must be treated separately when they are in motion with respect to each other, as Webster [25] pointed out (cf also [10,11]). The above resolution to the paradox seems to be more satisfactory than that proposed in Reference [18], involving another infinite wire.

The above argument implies the following S- and S'-scenarios. In S, in the original "no current" state, assuming overall neutrality, the ions and

electrons volume charge densities are ρ_+ and $-\rho_+$. Next, under the action of the axial electric field $\mathbf{E}_\parallel = E_\parallel(-\mathbf{u}_z)$, the free-electrons acquire the drift velocity $\mathbf{v} = v\mathbf{u}_z$ while their charge density is still $-\rho_+$ everywhere in that "current without pinch" state [5,18]. Eventually, the pinch effect develops under the action of the magnetic field of the current itself, charge separation occurs producing a transverse electric field, naked positive layer and final electrons charge density $-\rho_+\gamma^2$. On the other hand, in S', in "no current" state, the corresponding charge densities are $\rho_+\gamma$ and $-\rho_+\gamma$ everywhere, both ions and electrons (on average) travel to the left with velocity $-\mathbf{v}$, producing two *convection* currents equal in magnitude but of opposite directions. Next, under the action of an axial electric field $\mathbf{E}'_\parallel$ to the left, the electrons, on average, stop and remain at rest, now with charge density $-\rho_+/\gamma$ throughout the wire, and thus only the ions current to the left survives,

$$I' = \rho'_+\pi a^2 v = \gamma I \,, \tag{14}$$

using Equation (11). (The analogous asymmetry between the S- and S'-descriptions is found in the well-known "thread-between-spaceships" relativistic problem [26], cf also [27-30].) Eventually, the pinch effect develops under the action of the radial electric field directed outward, leading to final electrons charge density $-\rho_+\gamma$ in the cylindrical region ($r < a - \delta$) and thus to vanishing of the radial electric field inside the cylinder.

It should be stressed that the above distinction between "current without pinch" and "current with pinch" stages is somewhat simplistic; a more realistic scenario should take into account that the pinch effect develops at the same time as the current establishes in the wire. However, conceptual traps lurk even in the simplistic scenario.

Now we shall briefly discuss the electric and magnetic fields in S'.

In S, the constant axial electric field $\mathbf{E}_\parallel = E_\parallel(-\mathbf{u}_z)$ given by Equation (9) is needed to produce and keep the steady (conduction) current I to the left due to the free-electrons with drift velocity $\mathbf{v} = v\mathbf{u}_z$ to the right. On the other hand, in S', an axial electric field $\mathbf{E}'_\parallel$ is needed to annihilate the initial convection current of the free-electrons (due to their motion with velocity

$-\mathbf{v}$ to the left), and keep them, on average, at rest, despite their "home-lattice" remains in uniform motion with velocity $-\mathbf{v}$. Symmetry suggests that *everywhere*

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel}\,. \tag{15}$$

(Recall, however, that one should be careful with symmetries in Special Relativity [31,32], [27,28].) Equation (15) is consistent with the transformation law for $\mathbf{E}$ and $\mathbf{B}$

$$\begin{aligned} \mathbf{E}_{\parallel} &= \mathbf{E}'_{\parallel}\,, \qquad & \mathbf{E}_{\perp} &= \gamma[\mathbf{E}'_{\perp} - \mathbf{v}\times\mathbf{B}']\,, \\ \mathbf{B}_{\parallel} &= \mathbf{B}'_{\parallel}\,, \qquad & \mathbf{B}_{\perp} &= \gamma[\mathbf{B}'_{\perp} + (1/c^2)\mathbf{v}\times\mathbf{E}']\,, \end{aligned} \tag{16}$$

where $\parallel$ and $\perp$ denote the field components parallel and normal to $\mathbf{v}$, respectively [33,12]. Also, as can be seen, Equation (15) is consistent with contraption for producing $\mathbf{E}_{\parallel}$ in S, consisting of two parallel charged conducting planes at rest in S which are perpendicular to the wire [34,35].
[Note that a related problem of an infinitely long cylindrical wire carrying a steady current with cylindrically symmetric return path is usually considered without taking into account the self-induced pinch-effect. In that approximation, a correct analytic solution implies that there is a distribution of charge over the surface of the wire whose density is a *linear* function of the axial coordinate z, cf [36] and also [37]. However, as Sommerfeld pointed out, "the zero point of the charge remains undetermined since the point $z = 0$ can be fixed arbitrarily. We may eventually identify it with the "center" of the wire, which, for infinite length, also remains indefinite" [36]. In our opinion, Sommerfeld's statement implies that the linear function solution for the surface charge density is *physically* meaningless. As can be seen, the problem persists even if the pinch effect is taken into account.]

Eventually, using Equations (9) and (15), Ampère's law and Gauss' law we find $\mathbf{E}'_{\perp}$ and $\mathbf{B}'$.

The magnitude of $\mathbf{B}'$ is given by

$$B' = \begin{cases} \mu_0 \gamma I r / 2\pi a^2\,, & r \le a,\,, \\ \mu_0 \gamma I / 2\pi r\,, & r \ge a\,, \end{cases} \tag{17}$$

and the transverse (radial) electric field can be expressed as

$$\mathbf{E}'_\perp = \begin{cases} 0\,, & r \le a - \delta\,, \\ \gamma I[r^2 - (a-\delta)^2]\mathbf{u}_r / 2\pi\varepsilon_0 r a^2 v\,, & a - \delta \le r \le a\,, \\ \mu_0 \gamma I v \mathbf{u}_r / 2\pi r\,, & r \ge a\,, \end{cases} \tag{18}$$

using relation $\rho'_+ = \gamma\rho_+$ and Equations (7) and (11).

It can be easily verified that expressions (8)-(10) and (15), (17) and (18) for the electric and magnetic fields in the S and S' frames are consistent with the transformation law (16), as they should be. Also, the test charge $q > 0$ which is momentarily at rest in S, and has the instantaneous velocity $-\mathbf{v}$ at the same space-time point in S', would subsequently deflect towards the wire, as observed in S' too.

2. Concluding comments

The above analysis of charges and fields of an infinitely long conducting wire carrying a steady current in the lattice frame and the free-electrons frame provides another illustration of the mutual consistency of classical electromagnetism and Special Relativity. It should be stressed that our conclusions are essentially based on a purely mathematical fact that Maxwell's equations and equation of motion of a charged particle in the electromagnetic field

$$\frac{d}{dt}\left(\frac{m\mathbf{u}}{\sqrt{1 - u^2/c^2}}\right) = q(\mathbf{E} + \mathbf{u} \times \mathbf{B})\,,$$

where the mass m is a time independent Lorentz scalar, are covariant with respect to the Lorentz transformation, regardless of the value of the ratio v/c appearing in it. Thus French's exclamation near the end of his excellent book: "Who says relativity is important only for velocities comparable to that of light?" [5], in the context of an analysis similar to ours, seems to be somewhat exaggerated. However, this does not mean that we in any way doubt Special Relativity or its importance even at "creeping" velocities (cf, e.g., [31], [38-40]).

Also, the above analysis leads to two somewhat surprising insights: first, the concept of surface charge must not be introduced for every extremely thin layer and, second, wire segments in our model cannot be reified. While the first insight is not related directly to Special Relativity, the second one is in opposition to our Galilean instincts, showing clearly why it is hard to acquire a relativistic mentality. In addition to these conceptual traps, it should be pointed out that some of our starting assumptions appear to be questionable.

First point, in a copper wire there is about 10^{23} Cu^{+} ions/cm^3, and for $v = 1$mm/s using Equation (5) we obtain that roughly 1 more electrons than protons is found in each cubic centimeter of the copper wire. For a wire with a cross-sectional area of about 1mm^2, this amounts to one electron per meter of the wire, which presumably suggests breakdown of the continuum model in our case. (Note that the same trouble with second-order charges appears in a recent analysis of a rotating conducting sphere [41].)

Another difficult point is the standard assumption (usually introduced by *fiat*, cf, e.g. [7]) that the net charge per unit length of an infinite cylindrical wire carrying a steady current is zero in the lattice frame. (The assumption is closely related to the Clausius postulate, "a closed constant current in a stationary conductor exerts no force on stationary electricity" [42,43], [31], [38].) Zapolsky [18] gave a relativistic justification of the standard assumption in the framework of an elementary but nontrivial "procession" model of conductor carrying a steady current for both an infinite straight conducting wire and a thin circular wire. While his argument appears to be correct in the latter case, it seems to be problematic in the case of the infinite wire since the author starts from the assumption that a constant axial electric field is "turned on" along the infinite wire at the moment $t = 0$. While that assumption is often found in textbooks (cf, e. g., [44]), it is clearly incompatible with Special Relativity. (The "turning-on" of the external electric field is a rather intricate process. For some insights into the analogous "turning-off" physics cf, e.g., [45].) Stress, however, that Zapolsky is right that spacing between two electrons which are accelerated, starting simultaneously from rest, in a constant electric field always remains equal to their initial separation,

all with respect to the S frame [18].

On the other hand, some authors [46] seem to imply that the spacing between electrons contracts as they are accelerated and thus deduce that the electronic charge distribution contracts as it is accelerated up to the drift velocity v during the transient buildup of the current, all with respect to S, invoking the current sinks and sources (i. e. batteries) in order to bring the wire back into neutrality. However, since the starting assumption is wrong (as Zapolsky [18] pointed out), the argument is inconclusive. Note that the same problematic assumption - that of the Lorentz contraction of the electronic charge distribution as compared to its initial (zero drift velocity) distribution - is implicit in a recent paper by Brill *et al* [47]. Namely, the authors claim that a straight, infinite, current-carrying wire, modeled as consisting of two infinite lines of charge (the ions at rest in S and the electrons at rest in S'), is electrically neutral neither in S nor in S' but in the "middle frame" for the S and S' frames (whose speed is $(1-\gamma^{-1})c^2/v$ relative to both frames, as a simple calculation shows). As can be seen, their claim would be correct only if, in S, distances between adjacent positive ions at rest and adjacent electrons moving at the drift speed v were, say, d and $d\sqrt{1-v^2/c^2}$, respectively (the respective distances as measured in S' would be $d\sqrt{1-v^2/c^2}$ and d). Recall that the same assumption of electrical neutrality in the *middle frame* was essentially tacitly used in the *first* edition of Purcell's book [7], and also in [8]. The same assumption was recently revived in the case of a cylindrical wire by Folman [48].[1]

One last point. As the above argument shows, the standard assumption that the net charge per unit length is zero in the lattice frame S is essentially based on Zapolsky's insight that spacing between two electrons does not change with time if the electrons are accelerated starting simultaneously from rest in a constant electric field, all with respect to S. (As observed in S', *decelerations* of the two electrons neither start nor stop simultaneously, which

[1]It seems that the possibility that the wire is neutral in the middle frame for S and S', involving the Lorentz contraction of the electronic charge distribution relative to the lattice frame, must be ruled out on the basis that the free electrons are not connected to rigidly moving rods.

accounts for increase in their separation with respect to S'.) What if, instead of the standard assumption, we assume that the net charge per unit length is zero in the free-electrons frame S' not only when there is no current in the wire but also when it carries a steady current? Obviously, that would require that we somehow manage to decelerate simultaneously until they stop the two electrons with respect to S', and thus their separation would not change with time in S'. Then the ions and electrons volume charge densities would be $\rho_+\gamma$ and $-\rho_+\gamma$ throughout the wire also when there is current in it (i. e. when the free-electrons stop and remain at rest). No pinch develops because there is no transverse electric force (no transverse electric field) and no magnetic force (the electrons are at rest). The corresponding charge densities in S would be ρ_+ and $-\rho_+\gamma^2$ throughout the wire; as can be seen, no pinch develops in S either, because the transverse electric force and the magnetic force on the moving electrons cancel each other.

To summarize, it appears that the problem of in what frame is an infinite current-carrying wire neutral does not have a unique solution. It can be neutral either in the lattice frame S or in the electrons frame S', depending on the way the current is established in the wire (by a simultaneous acceleration of the electrons relative to S, or by a simultaneous deceleration of the electrons relative to S', respectively). Both cases are consistent with Special Relativity, since the corresponding equilibrium situations in the S and S' frames are described on the basis of classical electromagnetism, which is Lorentz covariant. Our simple results for equilibrium charge and current distributions satisfy Maxwell's equations. Thus, perhaps, the above analysis is not devoid of physical sense.

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